THE CAUCHY PROBLEM FOR A FIRST-ORDER ORDINARY LINEAR DIFFERENTIAL EQUATION WITH A VARIABLE COEFFICIENT AND CONTINUOUSLY VARYING ORDER OF THE DERIVATIVE

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Abstract. In the presented work, the Cauchy problem for a first-order ordinary inhomogeneous differential equation with a variable coefficient and continuously varying order of the derivative is considered and a new expression for Dirac's "Delta" function is given. Note that this function was introduced by physicists.

The stated Cauchy problem is reduced to the Volterra-type integral equation of the second kind and is solved by the successive approximation method. The convergence of the Neumann series was investigated using the resolvent.

Various problems for ordinary linear differential equations with a constant coefficient and continuously varying order of the derivative were discussed in [1-3].

In this work, the Cauchy problem is considered for the ordinary linear inhomogeneous differential equation with variable coefficient and continuously varying order of the derivative.

Euler algebraized the ordinary linear homogeneous equation with constant coefficients and gave an invariant function with respect to the derivative. Mittag-Lefler established this function for the ordinary linear fractional derivative equation with constant coefficients and for a linear equation with constant coefficients and continuously varying order of the derivative this function was established by V. Volterra [4].

Keywords: ordinary differential equation with a continuously varying order of the derivative, variable coefficient equation, inhomogeneous linear equation, Cauchy problem.

The statement of the problem

Let's consider the Cauchy problem as follows:

$$y'(x) - \int_0^1 D_{x_0^+}^{\alpha}\left[a(x,\alpha)y(x)\right]d\alpha = f(x), \quad x > x_{0,} \tag{1}$$

$$y(x_0) = y_0, \tag{2}$$

here $a(x,\alpha)$ is a continuous real-valued function of x and α with all its derivatives (with respect to x) up to the first order, x_0 and y_0 are marked real constants, and $f(x)$ is a continuous real-valued function. $D_{x_0^+}^{\alpha}[a(x,\alpha)y(x)]$ is the Riemann-Liouville derivative of α∈[0,1) order [5],

$$D_{x_0^+}^{\alpha}[a(x,\alpha)y(x)] = \frac{d}{dx}\int_{x_0}^{x}\frac{(x-t)^{-\alpha}}{(-\alpha)!}a(t,\alpha)y(t)dt, \tag{3}$$

and

$$(\alpha-1)!=\Gamma(\alpha)=\int_0^{\infty} e^{-t}\cdot t^{\alpha-1}dt, \tag{4}$$

is the Euler integral for the "gamma" function.

$$y'(x)-\int_0^1 d\alpha \frac{d}{dx}\int_{x_0}^{x}\frac{(x-t)^{-\alpha}}{(-\alpha)!}a(t,\alpha)y(t)dt=f(x)$$

$$y'(x)-\left(\int_0^1 d\alpha\int_{x_0}^{x}\frac{(x-t)^{-\alpha}}{(-\alpha)!}a(t,\alpha)y(t)dt\right)'=f(x).$$

Let's integrate the given equation (1) at (x_0, x):

$$y(x)-y(x_0)-\int_0^1 d\alpha\int_{x_0}^{x}\frac{(x-t)^{-\alpha}}{(-\alpha)!}a(t,\alpha)y(t)dt=\int_{x_0}^{x}f(t)dt,$$

$$y(x)-\int_0^1 d\alpha\int_{x_0}^{x}\frac{(x-t)^{-\alpha}}{(-\alpha)!}a(t,\alpha)y(t)dt=y_0+\int_{x_0}^{x}f(t)dt,$$

and we'll get the following:

$$y(x)=\int_{x_0}^{x}y(t)dt\int_0^1\frac{(x-t)^{-\alpha}}{(-\alpha)!}a(t,\alpha)d\alpha+\int_{x_0}^{x}f(t)dt+y_0. \tag{5}$$

Let's assume that

$$a(t,\alpha)=(-\alpha)!\,a(t), \tag{6}$$

then from (5) we get

$$y(x)=\int_{x_0}^{x}a(t)y(t)dt\int_0^1(x-t)^{-\alpha}d\alpha+\int_{x_0}^{x}f(t)dt+y_0, \tag{7}$$

As is known that

$$\int_0^1(x-t)^{-\alpha}d\alpha=-\int_0^1(x-t)^{-\alpha}d(-\alpha)=-\frac{(x-t)^{-1}-1}{\ln(x-t)}. \tag{8}$$

On the other hand, as mentioned after (8.20) in [5]:

$$\frac{x^{-1}}{(-1)!}=\delta(x)\,. \tag{9}$$

This shows that the Delta function is on the boundary between integrable (where the integral is continuous) and non-integrable functions. That is, the Delta function has an integral, but it is not continuous, it is a uniform Heaviside function. The step in N and Z is one, and the step in ℝ is (+0). The property of the delta function is (1-0). That's why

$$\frac{x^{-1}}{lnx}=\delta(x). \tag{10}$$

Then, if we consider (10) and (8), (7) will be in the following form.

$$y(x)=-\int_{x_0}^{x}a(t)y(t)dt\frac{(x-t)^{-1}-1}{\ln(x-t)}+\int_{x_0}^{x}f(t)dt+y_0,$$

$$y(x)=-\int_{x_0}^{x}a(t)y(t)\delta(x-t)dt+\int_{x_0}^{x}a(t)y(t)\frac{dt}{\ln(x-t)}+\int_{x_0}^{x}f(t)dt+y_0,$$

$$y(x) = -\frac{1}{2}a(x)y(x) + \int_{x_0}^{x} \frac{a(t)y(t)dt}{\ln(x-t)} + \int_{x_0}^{x} f(t)dt + y_0,$$

$$\left(1 + \frac{a(x)}{2}\right)y(x) = \int_{x_0}^{x} \frac{a(t)y(t)dt}{\ln(x-t)} + \int_{x_0}^{x} f(t)dt + y_0,$$

and within the condition

$$\left(1 + \frac{a(x)}{2}\right) \neq 0, \tag{11}$$

$$y(x) = \frac{2}{2+a(x)}\left[\int_{x_0}^{x} \frac{a(t)}{\ln(x-t)} y(t)dt + \int_{x_0}^{x} f(t)dt + y_0\right],$$

and

$$y(t) = \frac{2}{2+a(t)}\left[\int_{x_0}^{t} \frac{a(\tau)}{\ln(t-\tau)} y(\tau)d\tau + \int_{x_0}^{t} f(\tau)d\tau + y_0\right],$$

and according to the successive approximation method

$$y(x) = \frac{2}{2+a(x)}\left\{\left(\int_{x_0}^{x} \frac{a(t)dt}{\ln(x-t)}\left[\frac{2}{2+a(t)}\left(\int_{x_0}^{t} \frac{a(\tau)}{\ln(t-\tau)} y(\tau)d\tau + \right.\right.\right.\right.$$

$$\left.\left.\left.\left. + \int_{x_0}^{t} f(\tau)d\tau + y_0\right)\right] + \left(\int_{x_0}^{x} f(t)dt + y_0\right)\right)\right\},$$

$$y(x) = \frac{2}{2+a(x)}\int_{x_0}^{x} y(\tau)d\tau \int_{\tau}^{x} \frac{2a(t)a(\tau)dt}{(2+a(t))\ln(x-t)\ln(t-\tau))} +$$

$$+ \frac{2}{2+a(x)}\int_{x_0}^{x} \frac{a(t)dt}{\ln(x-t)} \frac{2}{2+a(t)}\int_{x_0}^{t} f(\tau)d\tau +$$

$$+ \frac{2}{2+a(x)}\int_{x_0}^{x} \frac{a(t)dt}{\ln(x-t)} \frac{2}{2+a(t)} y_0 + \frac{2}{2+a(x)}\int_{x_0}^{x} f(t)dt + \frac{2}{2+a(x)} y_0,$$

$$y(x) = \int_{x_0}^{x} y(\tau)d\tau \int_{\tau}^{x} \frac{2a(t)}{\big(2+a(x)\big)\ln(x-t)} \frac{2a(\tau)}{\big(2+a(t)\big)\ln(t-\tau)} dt +$$

$$+ \int_{x_0}^{x} \frac{2a\,(t)dt}{\big(2+a(x)\big)\ln(x-t)} \frac{2}{2+a(t)}\int_{x_0}^{t} f(\tau)d\tau +$$

$$+ \int_{x_0}^{x} \frac{2a(t)dt}{(2+a(x))\,ln(x-t)} \frac{2}{2+a(t)} y_0 + \frac{2}{2+a(x)}\left(\int_{x_0}^{x} f(t)dt + y_0\right). \tag{12}$$

The last equation can be written as follows:

$$y(x) = \int_{x_0}^{x} K_1(x,\tau)y(\tau)d\tau + F_1(x)\,. \tag{13}$$

So here the functions

$$K_1(x,\tau) = \int_{\tau}^{x} \frac{2a(t)}{\big(2+a(x)\big)\ln(x-t)} \frac{2a(\tau)}{\big(2+a(t)\big)\ln(t-\tau)} dt, \tag{14}$$

and

$$F_1(x)=\int_{x_0}^{x}\frac{2a(t)dt}{\big(2+a(x)\big)\ln(x-t)}\frac{2}{2+a(t)}\int_{x_0}^{t}f(\tau)d\tau+$$

$$+\int_{x_0}^{x}\frac{2a(t)dt}{\big(2+a(x)\big)\ln(x-t)}\frac{2}{2+a(t)}y_0+\frac{2}{2+a(x)}(\int_{x_0}^{x}f(t)dt+y_0), \qquad (15)$$

are continuous.

$$y(\tau)=\int_{x_0}^{\tau}K_1(\tau,z)y(z)dz+F_1(\tau)\ , \qquad (16)$$

and if we denote

$$y(x)=\int_{x_0}^{x}K_1(x,\tau)d\tau\left[\int_{x_0}^{\tau}K_1(\tau,z)y(z)dz+F_1(\tau)\right]+F_1(x)=$$

$$=\int_{x_0}^{x}y(z)dz\int_{z}^{x}K_1(x,\tau)K_1(\tau,z)d\tau+\int_{x_0}^{x}K_1(x,\tau)F_1(\tau)d\tau+F_1(x).$$

$$\int_z^x K_1(x,\tau)K_1(\tau,\eta)d\tau=K_2(x,\eta)\quad,\int_{x_0}^x K_1(x,\tau)F_1(\tau)d\tau+F_1(x)=F_2(x),$$

then

$$y(x)=\int_{x_0}^{x}K_2(x,\eta)y(\eta)d\eta+F_2(x). \qquad (17)$$

If we continue the process in this order, we get the Voltaire-type integral equations of the second kind in the following form

$$y(x)=\int_{x_0}^{x}K_n(x,t)y(t)dt+F_n(x)\,. \qquad (18)$$

It is easy to see that the kernel (14) is continuous under condition (11) and the right-hand side of (15) is bounded, i.e.

$$|K_1(x,t)|\le K_1, \qquad (19)$$

$$|F_1(x)|\le F_1, \qquad (20)$$

$$|K_2(x,\eta)|=\left|\int_{\eta}^{x}K_1(x,\tau)K_1(\tau,\eta)d\tau\right|\le {K_1}^2(x-\eta)\,, \qquad (21)$$

and we’ll get

$$|K_n(x,t)|\le {K_1}^n\frac{(x-t)^{n-1}}{(n-1)!}\,. \qquad (22)$$

If we go to the limit ($n\to\infty$) in (18), then

$$y(x)=\lim_{n\to\infty}F_n(x)=F_1(x)+\int_{x_0}^{x}K_1(x,t)F_1(t)dt+$$

$$+\int_{x_0}^{x}K_2(x,t)F_1(t)dt+\cdots+\int_{x_0}^{x}K_n(x,t)F_1(t)dt+\cdots,$$

$$R(x,t)=\sum_{m=1}^{\infty}K_m(x,t)\ , \qquad (23)$$

Then we’ll get

$$y(x) = F_1(x) + \int_{x_0}^{x} R(x,t)F_1(t)dt, \qquad (24)$$

The expression $R(x,t)$ is the resolvent of the kernel $K_1(x,t)$ and the series (23) is the Neumann series.

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TÖRƏMƏSİNİN TƏRTİBİ KƏSİLMƏZ DƏYİŞƏN BİRİNCİ TƏRTİB ADİ, XƏTTİ DƏYİŞƏN ƏMSALLI BİRCİNS OLMAYAN DİFERENSİAL TƏNLİK ÜÇÜN KOŞİ MƏSƏLƏSİ

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Təqdim olunan işdə törəməsinin tərtibi kəsilməz dəyişən birinci tərtib adi, xətti dəyişən əmsallı bircins olmayan diferensial tənlik üçün Koşi məsələsinə baxılmışdır və Dirakın "Delta" funksiyası üçün yeni ifadə verilmişdir. Qeyd edək ki, bu funksiya fiziklər tərəfindən daxil olunmuşdur.
Törəməsinin tərtibi kəsilməz dəyişən adi, sabit əmsallı, xətti diferensial tənliklər üçün müxtəlif məsələlərə [1-3] də baxılmışdır.

Qoyulmuş Koşi məsələsi ikinci növ Volterra tipli inteqral tənliyə gətirilərək, ardıcıl yaxınlaşma üsulu ilə həll edilmişdir. Rezolventa vasitəsi ilə alınan Neyman sırasının yığılması araşdırılmışdır.
Eyler adi, sabit əmsallı, xətti bircins tənliyi cəbriləşdirib, törəməyə nəzərən invariant funksiyanı verdiyi kimi, kəsr tərtib törəməli adi, sabit əmsallı xətti tənlik üçün bu funksiyanı Mittaq-Lefler yerinə yetirmişdir. Törəməsinin tərtibi kəsilməz dəyişən sabit əmsallı xətti tənlik üçün isə bu funksiyanı V. Volterra qurmuşdur [4].
Açar sözlər: törəməsinin tərtibi kəsilməz dəyişən adi differensial tənlik, dəyişən əmsallı tənlik, qeyri-bircins xətti tənlik, Koşi məsələsi

ЗАДАЧА КОШИ ДЛЯ НЕОДНОРОДНОГО, ОБЫКНОВЕННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ПЕРВОГО ПОРЯДКА С ПЕРЕМЕННЫМИ КОЭФФИЦИЕНТАМИ, ПОРЯДОК ПРОИЗВОДНОЙ КОТОРОГО МЕНЯЕТСЯ НЕПРЕРЫВНЫМ ОБРАЗОМ

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В представленной работе рассматривается задача Коши для неоднородного, обыкновенного дифференциального уравнения первого порядка с переменными коэффициентами, порядок производной которого меняется непрерывным образом и определено новое представление для δ-функции Дирака.
В работах [1-3] рассмотрены различные задачи для линейного обыкновенного дифференциального уравнения с постоянными коэффициентами, порядок производной которого, меняется непрерывным образом.
Рассматриваемая здесь задача Коши приводится к интегральному уравнению Вольтерра второго рода и решается с помощью метода последовательных приближений. Исследована сходимость ряда Неймана, полученной с помощью резольвенты.
Ейлер алгебраизировав обыкновенное, с постоянным коэффициентом линейное однородное уравнение, определил инвариантную функцию относительно производной. А Миттаг-Лефлер выполнил ту же самую работу для обыкновенного, с постоянным коэффициентом линейного уравнения с дробной производной. Для линейного дифференциального уравнения с постоянными коэффициентами, порядок производной, которого меняется непрерывным образом, эту функцию построил В.Вольтерра [4].
Ключевые слова: обыкновенное дифференциальное уравнение с непрерывно меняющимся порядком производной, уравнение с переменными коэффициентами, неоднородное линейное уравнение, задача Коши

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